Coherence and Phase-space III

VSSUP Lectures 2014

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January 22, 2014

Outline

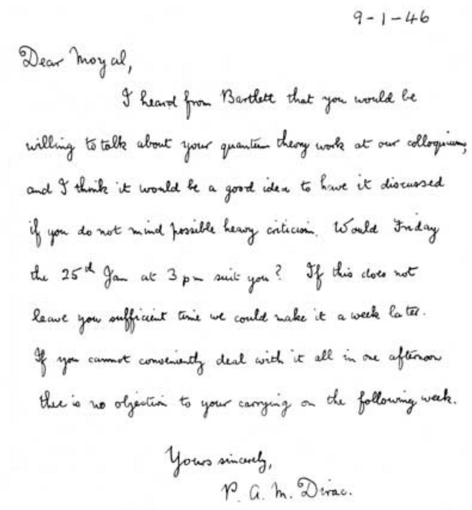
Dirac's objection to Wigner-Moyal phase-space

Moyal showed how to calculate time-evolution!

- Moyal brackets map quantum operators to differential equations
- **Correspondence with Dirac - who prevented publication!**
- Now widely used in many areas of physics and elsewhere

Dirac's criticism: ***probabilities can't have negative values***

Classical phase-space time-evolution

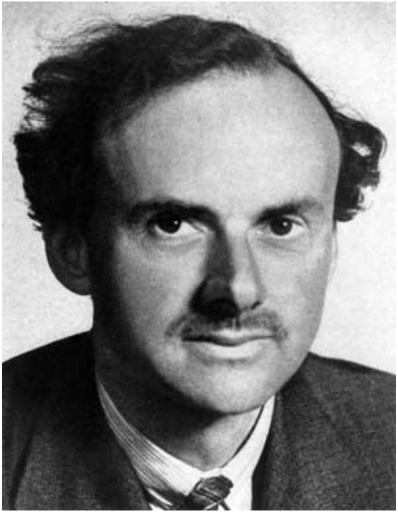


9-1-46

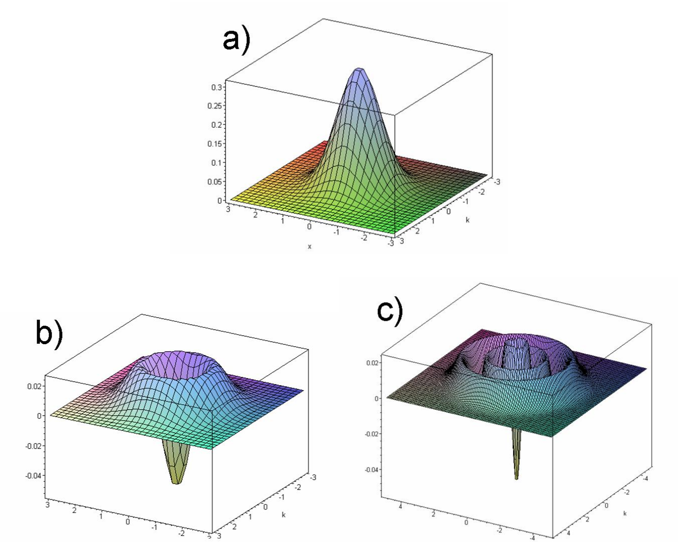
Dear Moyal,

I heard from Bartlett that you would be willing to talk about your quantum theory work at our colloquium, and I think it would be a good idea to have it discussed if you do not mind possible heavy criticism. Would Friday the 25th Jan at 3 pm suit you? If this does not leave you sufficient time we could make it a week later. If you cannot conveniently deal with it all in one afternoon there is no objection to your carrying on the following week.

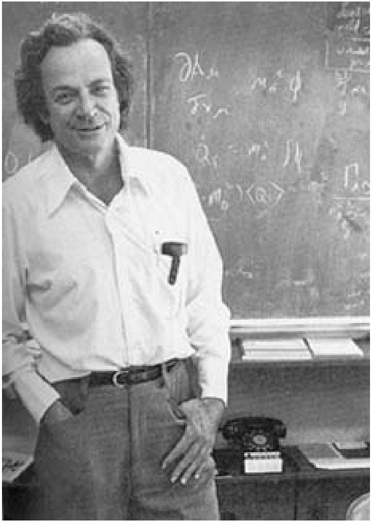
Yours sincerely,
P. A. M. Dirac.



Wigner distribution of number states- a:N=0, b:N=1, c:N=5



Feynman's 'Simulating Physics with Computers'



Simulating Physics with Computers

Richard P. Feynman

Department of Physics, California Institute of Technology, Pasadena, California 91107

Received May 7, 1981

1. INTRODUCTION

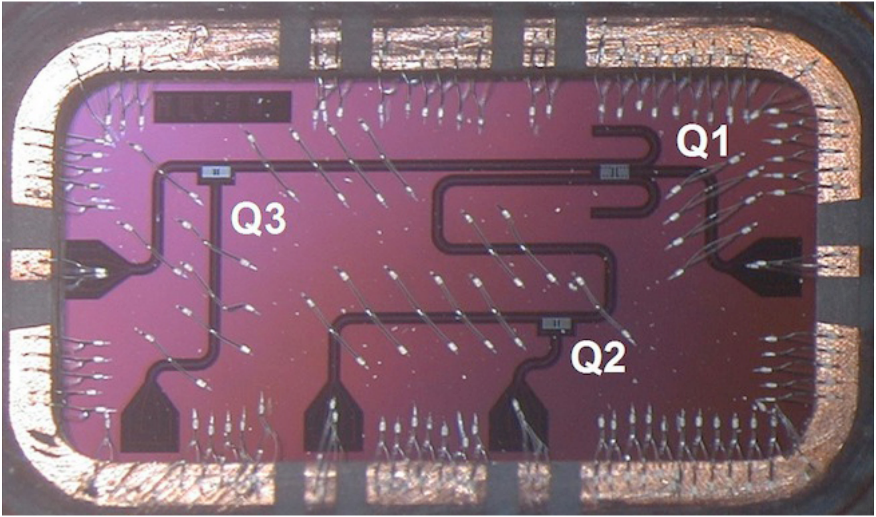
On the program it says this is a keynote speech—and I don't know what a keynote speech is. I do not intend in any way to suggest what should be in this meeting as a keynote of the subjects or anything like that. I have my own things to say and to talk about and there's no implication that anybody needs to talk about the same thing or anything like it. So what I want to talk about is what Mike Dertouzos suggested that nobody would talk about. I want to talk about the problem of simulating physics with computers and I mean that in a specific way which I am going to explain.

Feynman's argument

Can quantum systems be probabilistically simulated by a classical computer?

- 'In other words a computer which will give the same probabilities as the quantum system does.'
- **'If you take the computer to be the classical kind and there's no changes in any laws, and there's no hocus-pocus, the answer is certainly, No!'**
- 'This is called the hidden variable problem: it is impossible to represent the results of quantum mechanics with a classical universal device.'
- Feyman then proposed the quantum computer!

2012: IBM 3 qubit superconducting quantum computer



Is three qubits enough for simulations?

Quantum many-body problems are large even in qubits!

- consider N particles distributed among M modes
- take $N \simeq M \simeq 500,000$:
- Number of quantum states: $N_s = 2^{2N} = 2^{1,000,000}$
- **This is equivalent to one million qubits**
- **Up to a billion qubits with error-correction!**
- How BIG is your **quantum** computer?

How can we overcome Feynman's argument?

We can simulate correlations, not observations!

- A computer can **calculate** correlations any way we like
- **We only have to generate predictions**
- Suppose $\left\langle \hat{J}_A^{\theta} \hat{J}_B^{\theta} \right\rangle_Q = C + \int J_A^{\theta} J_B^{\theta} P\left(\vec{J}\right) d\vec{J}$
- J_A^{θ} is a real or complex variable, C is an offset
- It doesn't matter if a cat is black or white, so long as it catches mice! (Deng Xiaoping)
- **It doesn't matter if a computer is quantum or classical, so long as it calculates measurements!**

Husimi's Q-function

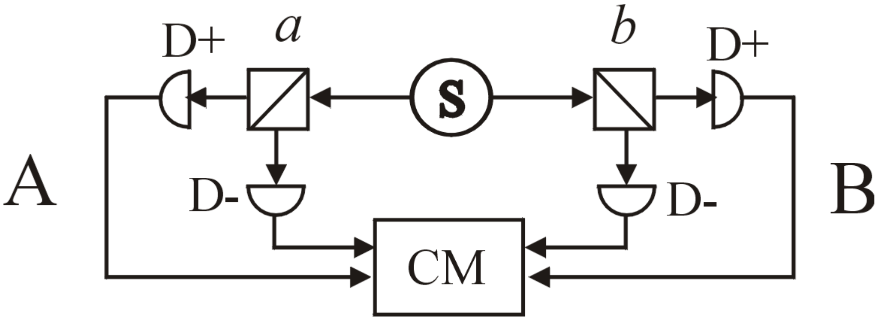
Husimi's Q-function is a positive phase-space method!

- consider coherent states $|\boldsymbol{\alpha}\rangle$ of M modes
  - Define $Q(\boldsymbol{\alpha}) = \langle\boldsymbol{\alpha}|\,\hat{\rho}\,|\boldsymbol{\alpha}\rangle/\pi^M$
  - Quantum correlations: $\left\langle \hat{a}_A^\dagger \hat{a}_B \right\rangle_Q = -\delta_{AB} + \int \alpha_A^* \alpha_B Q(\boldsymbol{\alpha})\, d^M\boldsymbol{\alpha}$
  - Problem - time-evolution is not easily computed

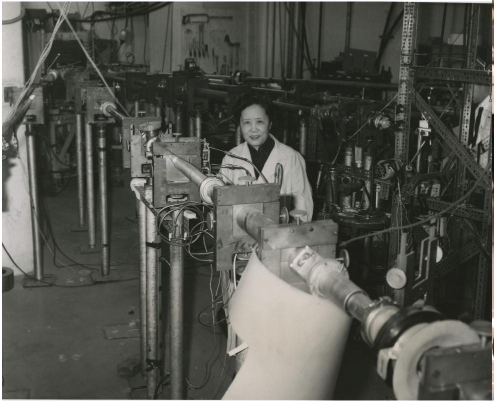
Exercise 1: show that for an N-boson number state,

$$Q(\alpha) \propto |\alpha|^{2N} \exp\left(-|\alpha|^2\right)$$

Can we represent a Bell state?



Wolf prize-winners: Wu (1978) and Aspect (2010)



Q-function with Bell inequality!

What is the Q-function of a Bell state?

- consider typical 4-mode Bell state of photons or atoms:
  - Define $|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle_{a+}|0\rangle_{a-}|1\rangle_{b+}|0\rangle_{b-} + |0\rangle_{a+}|1\rangle_{a-}|0\rangle_{b+}|1\rangle_{b-}\right]$
  - $Q(\boldsymbol{\alpha}) = |\langle\boldsymbol{\alpha}|\,\Psi\rangle|^2/\pi^M$

$$Q_B(\boldsymbol{\alpha}) \propto \left[|\alpha_+|^2|\beta_+|^2 + |\alpha_-|^2|\beta_-|^2\right]\exp\left(-|\boldsymbol{\alpha}|^2\right)$$

Exercise 2: Show that for the Bell inequality, one obtains $B = 2\sqrt{2}$ using the Q-function

Glauber's P-function

2005 Nobel Prize in Physics

- one half to **Roy J. Glauber**
  - ***for his contribution to the quantum theory of optical coherence***
- one half to **Ted Haensch** and **Jan Hall**
  - **for their contributions to the the development of laser-based precision spectroscopy**

Glauber and Sudarshan's P-function

Glauber's P-function is a normally-ordered phase-space method!

- consider coherent states $|\boldsymbol{\alpha}\rangle$ of M modes
  - Define $P(\boldsymbol{\alpha})$ implicitly:
  - $\hat{\rho} = \int P(\boldsymbol{\alpha}) |\boldsymbol{\alpha}\rangle \langle\boldsymbol{\alpha}| d^M\boldsymbol{\alpha}$
  - Quantum correlations: $\left\langle \hat{a}_A^\dagger \hat{a}_B \right\rangle_P = \int \alpha_A^* \alpha_B P(\boldsymbol{\alpha}) d^M\boldsymbol{\alpha}$
  - All correlations can be calculated IF $P(\boldsymbol{\alpha})$ exists
  - Problem: $P(\boldsymbol{\alpha})$ highly singular for nonclassical states

First-principle simulations

What do we do with modes having low occupation numbers?

- Truncated Wigner only works if all modes are heavily occupied
- P-function is singular for nonclassical states
- Q-function exists, but has no simple time-evolution
- How about modeling other cases with low occupations:
  - the formation of a BEC must start with low occupation!
  - collisions that generate atoms in initially empty modes
  - coupling to thermal modes having low occupation?
- We need a technique without the large N approximation

+P PHASE-SPACE METHODS

The positive P-representation is an expansion in coherent state projectors

$$\widehat{\rho} = \int P(\boldsymbol{\alpha},\boldsymbol{\beta})\widehat{\Lambda}(\boldsymbol{\alpha},\boldsymbol{\beta})d^{2M}\boldsymbol{\alpha}\,d^{2M}\boldsymbol{\beta}$$

$$\widehat{\Lambda}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\beta}^*|}{\langle\boldsymbol{\beta}^*|\,|\boldsymbol{\alpha}\rangle}$$

Enlarged phase-space allows positive probabilities!

- Maps quantum states into 4M real coordinates: $\boldsymbol{\alpha},\boldsymbol{\beta} = \mathbf{p}+i\mathbf{x},\mathbf{p}'+i\mathbf{x}'$
- Exact mappings even for low occupations
- Advantage: Can represent entangled states

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+P Existence Theorem

For ANY density matrix, a positive P-function always exists

$$P(\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{1}{(2\pi)^{2M}} e^{-\left|\boldsymbol{\alpha}-\boldsymbol{\beta}^*\right|^2/4} \left\langle \frac{\boldsymbol{\alpha}+\boldsymbol{\beta}^*}{2} \middle| \widehat{\rho} \middle| \frac{\boldsymbol{\alpha}+\boldsymbol{\beta}^*}{2} \right\rangle$$
$$\propto e^{-\left|\boldsymbol{\alpha}-\boldsymbol{\beta}^*\right|^2/4} Q\left(\frac{\boldsymbol{\alpha}+\boldsymbol{\beta}^*}{2}\right)$$

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- Advantage: Probabilistic sampling is possible
- Problem: Non-uniqueness allows sampling error to grows with time (chaotic)

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Operator identities

Differentiating the projection operator gives the following identities

$$
\begin{aligned}
\widehat{a}_n^{\dagger}\widehat{\rho} &\rightarrow \left[\beta_n - \frac{\partial}{\partial \alpha_n}\right] P \\
\widehat{a}_n\widehat{\rho} &\rightarrow \alpha_n P \\
\widehat{\rho}\,\widehat{a}_n &\rightarrow \left[\alpha_n - \frac{\partial}{\partial \beta_n}\right] P \\
\widehat{\rho}\,\widehat{a}_n^{\dagger} &\rightarrow \beta_n P
\end{aligned}
$$

Since the projector is an analytic function of both α_n and β_n, we can obtain alternate identities by replacing $\partial/\partial\alpha$ by either $\partial/\partial\alpha_x$ or $\partial/i\partial\alpha_y$. This equivalence allows a positive-definite diffusion to be obtained, with stochastic evolution.

Measurements

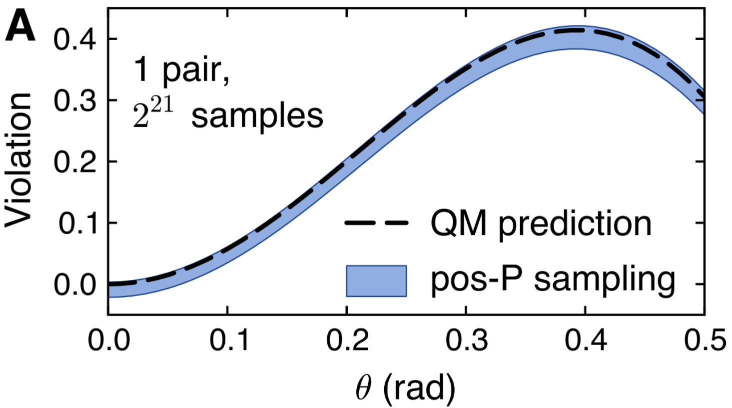
How do we calculate an operator expectation value

- There is a correspondence between the moments of the distribution, and the normally ordered operator products.
- These come from the fact that coherent states are eigenstates of the annihilation operator
- Using $\mathrm{Tr}\left[\widehat{\Lambda}(\boldsymbol{\alpha},\boldsymbol{\beta})\right] = 1$:

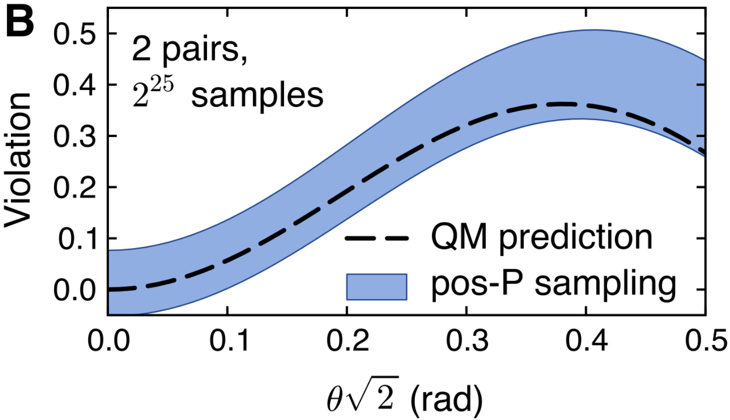
$$\langle \widehat{a}_m^{\dagger} \cdots \widehat{a}_n \rangle = \int \int P(\boldsymbol{\alpha},\boldsymbol{\beta})[\beta_m \cdots \alpha_n] d^{2M}\boldsymbol{\alpha}\, d^{2M}\boldsymbol{\beta}\,.$$

Exercise 3: Derive the moment correspondences from the definition of the positive P-function

Two particle Bell inequalities (Clauser/Aspect)



Four particle Bell inequalities (Bouwmeester)



Example: time-evolution of harmonic oscillator

Consider the harmonic oscillator

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$$

$$\frac{\partial \hat{\rho}}{\partial t} = -i\omega \left[\hat{a}^\dagger \hat{a}\widehat{\rho} - \widehat{\rho}\hat{a}^\dagger \hat{a}\right]$$

Operator mappings

- $\hat{a}^\dagger \hat{a}\hat{\rho} \rightarrow \left[\beta - \frac{\partial}{\partial\alpha}\right]\alpha P$
- $\hat{\rho}\hat{a}^\dagger \hat{a} \rightarrow \left[\alpha - \frac{\partial}{\partial\beta}\right]\beta P$
- $$\frac{\partial P}{\partial t} = i\omega\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\beta}\beta\right) P$$

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- $$\frac{\partial P}{\partial t} = i\omega\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\beta}\beta\right)P$$

Harmonic oscillator solution

General result for harmonic oscillator

$$\frac{\partial P}{\partial t} = i\omega\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\beta}\beta\right)P$$

Solution by method of characteristics, for initial delta function

- $$\frac{d\alpha}{dt} = -i\omega\alpha \implies \alpha(t) = \alpha(0)e^{-i\omega t}$$
- $P(\alpha,t) = \delta(\alpha - \alpha(t))$

Exercise 4: Prove that the characteristic function is the solution

General case

Suppose we have a more general Hamiltonian, like the BEC case. Then we define

$$\overrightarrow{\alpha} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$$

and find using operator mappings that - provided the distribution is sufficiently bounded at infinity:

$$\frac{\partial}{\partial t} P(t, \overrightarrow{\alpha}) = \left[\partial_i A_i(\overrightarrow{\alpha}) + \frac{1}{2} \partial_i \partial_j D_{ij}(t, \overrightarrow{\alpha}) \right] P(t, \overrightarrow{\alpha}) \,.$$

Comparison of positive-P and Wigner

- No other terms in +P - higher order derivatives all vanish
- Nonlinear couplings cause noise, linear damping does not

SUMMARY

Nonclassical phase-space representation methods are useful

+P phase-space is relatively simple!

- Maps quantum field evolution to a stochastic equation
- Can be used to treat complex systems in 3D
- Advantages: No truncation, no exponential complexity issues!
- Mathematical challenge:
  - sampling error increases in time

SUMMARY

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